

Standard decomposition of expansive ergodically supported dynamics

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Abstract

In this work we introduce the notion of weak quasi groups, that are, quasi-group operations defined almost everywhere on some set. Then we present sufficient conditions for an expansive ergodic map $T : X \rightarrow X$ to be an automorphism for some topological weak quasi group. Therefore, we find out an Abelian topological weak group operation and a standard decomposition of the dynamics of T in terms of T -invariant weak sub-groups.

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1 Introduction

The problem of characterizing the dynamical behavior of maps which are endomorphisms for compact groups have been widely studied in the last years. One of the first works on this subject is due to R. Bowen [2], who studied the entropy of such maps and showed that the Haar measure is the maximum entropy measure for certain class of algebraic dynamical systems. Latter, in [4], D. Lind proved that

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ergodic maps which are automorphisms for compact Abelian groups are always conjugated to some full shift. For the case where $(X, +)$ is any topological group with X being a zero-dimensional space, B. Kitchens [3] proved that any expansive endomorphism $T : X \rightarrow X$ can be represented as a shift map defined on the cartesian product of a full shift with a finite set. In [6], this result was extended for expansive maps which are endomorphisms for certain class of zero-dimensional quasi groups.

We say (X, T) is a *topological dynamical system* if X is a compact metric space and $T : X \rightarrow X$ is a continuous onto map. The topological entropy of (X, T) will be denoted by $\mathbf{h}(T)$. We say (X, T) and (Y, S) are *conjugated* if there exists an invertible map $\mathbf{f} : X \rightarrow Y$ such that $\mathbf{f} \circ T = S \circ \mathbf{f}$. In the case when \mathbf{f} is a homeomorphism we say (X, T) and (Y, S) are *topologically conjugated*.

Given a finite alphabet \mathcal{A} , define $\mathcal{A}^{\mathbb{S}} := \{(x_i)_{i \in \mathbb{S}} : x_i \in \mathcal{A}, \forall i \in \mathbb{S}\}$, with $\mathbb{S} = \mathbb{Z}$ or $\mathbb{S} = \mathbb{N}$. We consider in $\mathcal{A}^{\mathbb{S}}$ the product topology which is generated by the clopen subsets called cylinders. Let $\sigma_{\mathcal{A}^{\mathbb{S}}} : \mathcal{A}^{\mathbb{S}} \rightarrow \mathcal{A}^{\mathbb{S}}$ be the *shift map* defined by $\sigma_{\mathcal{A}^{\mathbb{S}}}((x_i)_{i \in \mathbb{S}}) = (x_{i+1})_{i \in \mathbb{S}}$. Therefore, a symbolic dynamical system is a topological dynamical system $(\Lambda, \sigma_{\Lambda})$ where $\Lambda \subseteq \mathcal{A}^{\mathbb{S}}$ is a closed subset such that $\sigma_{\mathcal{A}^{\mathbb{S}}}(\Lambda) = \Lambda$, and σ_{Λ} is the restriction of $\sigma_{\mathcal{A}^{\mathbb{S}}}$ to Λ (in this case we refer Λ as a *shift space*). A special type of shift spaces are the *Markov shifts*, which are those symbolic dynamical systems that can be constructed from walks on finite directed graphs (see [5] for more details).

A topological dynamical system (X, T) , is said to be *expansive* if there exists a family $\{U_i\}_{1 \leq i \leq k}$ of open sets, such that $\overline{\cup_{1 \leq i \leq k} U_i} = X$ and for $\mathbf{x}, \mathbf{y} \in X$, $\mathbf{x} \neq \mathbf{y}$, there exists $1 \leq i \leq k$ and $n \in \mathbb{S}$ (with $\mathbb{S} = \mathbb{Z}$ if T is invertible and $\mathbb{S} = \mathbb{N}$ otherwise) for that $T^n(\mathbf{x}) \in \bar{U}_i$ and $T^n(\mathbf{y}) \notin \bar{U}_i$. When (X, T) is expansive we can define its symbolic representation as the shift space $(\Lambda, \sigma_{\Lambda})$, where $\Lambda \subseteq \{1, \dots, k\}^{\mathbb{S}}$ is such that $(q_i)_{i \in \mathbb{S}} \in \Lambda$ if and only if there exists $\mathbf{x}_0 \in X$ such that for all $i \in \mathbb{S}$ we have $T^i(\mathbf{x}_0) \in U_{q_i}$.

We say (X, T) has the *shadowing property* if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $(\mathbf{y}_n)_{n \in \mathbb{S}} \in X$ is a sequence which verifies $d(T(\mathbf{y}_n), \mathbf{y}_{n+1}) < \delta$ for all $i \in \mathbb{S}$, then there exists $\mathbf{x}_0 \in X$ such that $d(T^n(\mathbf{x}_0), \mathbf{y}_n) < \epsilon$.

Given (X, T) , define $X^T := \{(x_i)_{i \in \mathbb{Z}} : x_{i+1} = T(x_i), \forall i \in \mathbb{Z}\}$. It is

well known that there exists a standard metric on X^T which makes it compact, and for which the shift map $\sigma_T : X^T \rightarrow X^T$ is continuous (see [7]). The topological dynamical system (X^T, σ_T) is called the *inverse limit system* of (X, T) . The projection $p : X^T \rightarrow X$ which takes the sequence $(x_i)_{i \in \mathbb{Z}}$ to x_0 is continuous and commutes with the maps σ_T and T , that is, $p \circ \sigma_T = T \circ p$. In particular, if T is invertible, then p also is invertible, and since T is continuous then p^{-1} is also continuous. Therefore, in such a case, p is a topological conjugation of (X, T) and (X^T, σ_T) .

A probability measure on X is said to be an *ergodically supported measure* if it is an ergodic measure which assigns positive measure for any nonempty open subset of X . Thus, we say (X, T) is *ergodically supported* if there exists an ergodically supported measure for it. A set $E \subset X$ which has zero measure for any ergodically supported measure is said a *universally null set*.

If for all $n \geq 1$ we have that (X, T^n) is ergodically supported, then we say (X, T) is *ergodically aperiodic*. On the other hand, we say (X, T) has *ergodic period* B if (X, T) is ergodically supported and there exists a finite family $\{C_i\}_{0 \leq i \leq B-1}$ of closed sets, such that: $X = \bigcup_{i=0}^{B-1} C_i$; $C_i \cap C_j$ is universally null set for any $i \neq j$; $T(C_i) = C_{i+1 \pmod{B}}$; and (C_i, T^B) is ergodically aperiodic for all i .

An important concept in dynamical system is the *almost topological conjugacy* of two dynamical systems. Such a concept was introduced by R. Adler and B. Marcus in [1] to study invariants of Markov shifts and latter extended by W. Sun in [7] to dynamical systems whose symbolic representations are Markov shifts. Due to the central role played by almost topological conjugacies in this work, we present below its definition due to Sun.

Definition 1.1. *Two ergodically supported topological dynamical systems (X, T) and (Y, S) are said almost topologically conjugate if there exist an ergodically supported Markov shift $(\Lambda, \sigma_\Lambda)$ and two continuous onto maps $\mathbf{f}_T : \Lambda \rightarrow X^T$ and $\mathbf{f}_S : \Lambda \rightarrow Y^S$ such that:*

$$(i) \quad \sigma_{X^T} \circ \mathbf{f}_T = \mathbf{f}_T \circ \sigma \quad \text{and} \quad \sigma_{Y^S} \circ \mathbf{f}_S = \mathbf{f}_S \circ \sigma;$$

(ii) *There exist a σ_{X^T} -invariant universally null set $M_2 \subset X^T$ and a σ_{Y^S} -invariant universally null set $P_2 \subset Y^S$, such that $\mathbf{f}_T : \Lambda \setminus M_1 \rightarrow$*

$X^T \setminus M_2$ and $\mathbf{f}_S : \Lambda \setminus P_1 \rightarrow Y^S \setminus P_2$ are one-to-one, where $M_1 = \mathbf{f}_T^{-1}(M_2)$ and $P_1 = \mathbf{f}_S^{-1}(P_2)$.

In this work, we prove that the topological entropy and the ergodic period of an invertible expansive ergodically-supported dynamical system with the shadowing property (X, T) establishes a sufficient criterion for the existence of quasi-group operations defined almost everywhere outside of universally null sets and for what T is an automorphism (Theorem 3.4). As consequence of this result, we prove that if (X, T) is aperiodic and has topological entropy $\log(N)$ for N integer, then we can find an Abelian operation defined almost everywhere and decompose the dynamics of T in terms of a finite family of sub-groups (Theorem 3.5). In this way we obtain for ergodic maps an analogous to the decomposition of linear maps in terms of their eigenspaces.

2 Quasi groups and weak quasi groups

Let G be a set and let $*$ be a binary operation on G . We say that $*$ is a *quasi-group operation* if $*$ is left and right cancelable. If, in addition, G is a topological space and $*$ is continuous, we say that $*$ is a *topological quasi-group operation*. For the case when G is finite the quasi-group operation $*$ is associated to a Latin square. Furthermore, it is easy to check that if for any $g \in G$ it follows that $g * G = G$ (which always occurs if G is finite), then $*$ is an associative quasi-group operation if and only if $*$ is a group operation.

For a finite set G , we will say that $s : G \rightarrow G$ is a translation on G , if given any $x \in G$ we have that $G = \{s^k(x) : k = 0, \dots, \#G - 1\}$, where $\#G$ denotes the cardinality of G . The following theorem gives a sufficient and necessary condition on the cardinality of a finite set G for the existence of quasi-group operations on G for what a given translation is an automorphism.

Lemma 2.1. *Given a finite set G and a translation $s : G \rightarrow G$, there exists a quasi group operation $*$ on G for which s is an automorphism if and only if G has a odd quantity of elements.*

Proof. Let $n := \#G$ and denote as $\tilde{+}$ the sum *mod* n . Note that, without loss of generality, we can consider $G := \{0, \dots, n-1\}$ and the translation on G in the form

$$s(x) = x\tilde{+}1.$$

If n is odd, we can define $\lambda := (n+1)/2$ and, since $\gcd(\lambda, n) = 1$ we have that the binary operation $*$ defined for all $x, y \in G$ by

$$x * y := \lambda(x\tilde{+}y),$$

where λz stands for z summed λ times with itself (*mod* n). Therefore, for any $x, y \in G$ we have that

$$\begin{aligned} s(x) * s(y) &= (x\tilde{+}1) * (y\tilde{+}1) = \lambda[(x\tilde{+}1)\tilde{+}(y\tilde{+}1)] = \\ &= \lambda(x\tilde{+}y\tilde{+}2) = \lambda(x\tilde{+}y)\tilde{+}\lambda 2 = (x * y)\tilde{+}1 = s(x * y). \end{aligned}$$

Now, let us show that if n is even, then it is not possible to exit a quasi-group operation for which s is automorphism. For this, consider that $*$ is some binary operation on G for which s is an automorphism. We can represent the action of $*$ on G by a table where the entry in the row indexed by x and column indexed by y represents the product $x * y$. Note that since s is an automorphism for $*$, then the table shapes as follows:

*	0	1	2	...	n-1
0	a_0	a_1	a_2	\dots	a_{n-1}
1	$a_{n-1}\tilde{+}1$	$a_0\tilde{+}1$	$a_1\tilde{+}1$	\ddots	$a_{n-2}\tilde{+}1$
2	$a_{n-2}\tilde{+}2$	$a_{n-1}\tilde{+}2$	$a_0\tilde{+}2$	\ddots	$a_{n-3}\tilde{+}2$
\vdots	\vdots	\ddots	\ddots	\ddots	\vdots
n-1	$a_1\tilde{+}(n-1)$	$a_2\tilde{+}(n-1)$	$a_3\tilde{+}(n-1)$	\dots	$a_0\tilde{+}(n-1)$

where $a_0, a_1, \dots, a_{n-1} \in \{0, \dots, n-1\}$. Hence, for all $x, y \in G$, we can write $x * y = a_{y\tilde{+}(n-x)}\tilde{+}x$. We recall that $*$ was a quasi-group operation if and only if the above table was a Latin square, that is, if and only if there was not repetition of elements in any row and any column of the table. Therefore, the sum *mod* n of all elements of any row or of all elements of any column should result in the same value. But, if

n is even, supposing by contradiction that there is no repetitions in any row of the table (that is, $a_i \neq a_j$, for $i \neq j$), we get that the sum $\text{mod } n$ over any row is

$$\sum_{y=0}^{n-1} x * y = \sum_{y=0}^{n-1} (a_{y \tilde{+} (n-x)} \tilde{+} x) = \frac{1}{2}(n-1)n \tilde{+} nx \pmod{n} = \frac{n}{2},$$

while the sum \pmod{n} over any column is

$$\sum_{x=0}^{n-1} x * y = \sum_{x=0}^{n-1} (a_{y \tilde{+} (n-x)} \tilde{+} x) = \frac{1}{2}(n-1)n \tilde{+} \frac{1}{2}(n-1)n \pmod{n} = 0,$$

which contradicts that there are not repetitions in any row or any column. \square

In the next section we shall look for quasi-group operations, for which a given dynamical system is an automorphism. However, if (X, T) is ergodic with nonzero entropy, then, in general, there does not exist such a quasi-group operation (with exception for dynamical systems on zero-dimensional spaces). In fact, to the general case, we will need some ‘weakness’ in the operation.

Definition 2.2. *Given a probability measure μ on the Borelians of G , we will say $*$ is a weak-quasi group operation with respect to μ if $*$ is a quasi-group operation which is well defined for $\mu \times \mu$ -almost all $(a, b) \in G \times G$. If in addition the operation $*$ is continuous on its domain, then we will say it is a topological weak quasi-group operation with respect to μ . When $*$ is a weak quasi-group operation with respect to μ on G , we will call $(G, *, \mu)$ a (topological) weak quasi group. Furthermore, if a (topological) weak quasi group is associative, then we will simply say it is a (topological) weak group*

Note that the definition of a weak quasi-group operation is made on the product space $G \times G$ and not on the space G . Thus, it is possible that there exist $x \in G$ and a null measure subset of $A \subset G$ such that for all $x \in A$ the products $x * y$ and $y * x$ are not defined for any $y \in G$.

On the other hand, if $*$ is a weak quasi-group operation with respect to some measure μ on G , then given $x, y \in G$, the existence

of the product $x * y$ does not imply the existence of $y * x$ (but when $*$ is commutative). Furthermore, the cancelable property of a weak quasi-group operation $*$ means that if $x * y$ (or $y * x$) and $x * z$ (or $z * x$) are defined, then $x * y = x * z$ (or $y * x = z * x$) if and only if $y = z$. In the same way, the associativity of a weak group holds only when both $x * (y * z)$ and $(x * y) * z$ are defined.

3 Weak quasi groups and expansive ergodically supported automorphisms

In order to construct a topological weak quasi group for which a given topological dynamical system is automorphism, we need the following results.

Lemma 3.1. *Let $\alpha : \mathcal{X} \rightarrow \mathcal{Y}$ be a continuous and onto map between topological spaces, and suppose \mathcal{X} is compact and \mathcal{Y} is Hausdorff. Given $\tilde{\mathcal{Y}} \subseteq \mathcal{Y}$, define $\tilde{\mathcal{X}} := \alpha^{-1}(\tilde{\mathcal{Y}})$. If the restriction $\tilde{\alpha} : \tilde{\mathcal{X}} \rightarrow \tilde{\mathcal{Y}}$ is one-to-one, then $\tilde{\alpha}$ is a homeomorphism.*

□

The proof of the above result is direct and it is left to the reader. Hence, by using the above lemma we can prove that two invertible dynamical systems that are almost topologically conjugated are topologically conjugated outside of universally null sets:

Theorem 3.2. *If two invertible dynamical systems (X, T) and (Y, S) are almost topologically conjugated, then there exists a homeomorphism $\varphi : \tilde{X} \rightarrow \tilde{Y}$ between $\tilde{X} \subseteq X$ and $\tilde{Y} \subseteq Y$ total-measure subsets with respect to any ergodically supported measure, which is a topological conjugacy between (\tilde{X}, T) and (\tilde{Y}, S)*

Proof. Let (X^T, σ_{X^T}) and (Y^S, σ_{Y^S}) be the inverse limit systems of (X, T) and (Y, S) , respectively. Note that since both (X, T) and (Y, S) are invertible, then the projections $p_T : X^T \rightarrow X$ and $p_S : Y^S \rightarrow Y$ are homeomorphism.

Let (Σ, σ) , $\mathbf{f}_T : \Sigma \rightarrow X^T$ and $\mathbf{f}_S : \Sigma \rightarrow Y^S$, be the Markov shift and the maps given in the definition of almost topological conjugacy. Also denote as $M_2 \subseteq X^T$ and $P_2 \subseteq Y^S$, and as $M_1 := \mathbf{f}_T^{-1}(M_2)$ and $P_1 := \mathbf{f}_S^{-1}(P_2)$, the universally null sets which make the maps

$\mathbf{f}_T : \Sigma \setminus M_1 \rightarrow X^T \setminus M_2$ and $\mathbf{f}_S : \Sigma \setminus P_1 \rightarrow Y^S \setminus P_2$ to be bijections. Denote as $\bar{\mathbf{f}}_T$ and $\bar{\mathbf{f}}_S$ these restrictions of \mathbf{f}_T to $\Sigma \setminus M_1$ and of \mathbf{f}_S to $\Sigma \setminus P_1$, respectively. From Lemma 3.1, we get that $\bar{\mathbf{f}}_T$ and $\bar{\mathbf{f}}_S$ are homeomorphisms.

Note that since p_T and p_S are homeomorphisms, the sets $M_3 := p_T(M_2)$ and $P_3 := p_S(P_2)$ are also universally null sets. Denote as \bar{p}_T and as \bar{p}_S the restrictions $p_T : X^T \setminus M_2 \rightarrow X \setminus M_3$ and $p_S : Y^S \setminus P_2 \rightarrow Y \setminus P_3$.

Thus, the maps $\gamma_T : \Sigma \setminus M_1 \rightarrow X \setminus M_3$ and $\gamma_S : \Sigma \setminus P_1 \rightarrow Y \setminus P_3$ defined by $\gamma_T := \bar{p}_T \circ \bar{\mathbf{f}}_T$ and $\gamma_S := \bar{p}_S \circ \bar{\mathbf{f}}_S$ are also homeomorphisms.

Note that $\tilde{\Sigma} := \Sigma \setminus (M_1 \cup P_1)$, is a total-measure subset of $\Sigma \setminus M_1$ and of $\Sigma \setminus P_1$, with respect to any ergodically supported measure. Hence, $\tilde{X} := \gamma_T(\tilde{\Sigma}) \subseteq X$ and $\tilde{Y} := \gamma_S(\tilde{\Sigma}) \subseteq Y$ are total-measure subsets with respect to any ergodically supported measure. Therefore we can consider $\tilde{\gamma}_T : \tilde{\Sigma} \rightarrow \tilde{X}$ and $\tilde{\gamma}_S : \tilde{\Sigma} \rightarrow \tilde{Y}$ the restrictions of γ_T and γ_S , respectively.

Finally, we define the homeomorphism $\varphi : \tilde{X} \rightarrow \tilde{Y}$ given by

$$\varphi := \tilde{\gamma}_S \circ \tilde{\gamma}_T^{-1}.$$

Since, all maps involved in the definition of φ commute with the dynamical systems, we get that φ is a topological conjugacy between (\tilde{X}, T) and (\tilde{Y}, S) . □

Corollary 3.3. *Let (X, T) and (Y, S) be two topological dynamical systems, and assume they are invertible, expansive, ergodically supported, have the shadowing property, and have the same ergodic period. Then, there exist $\tilde{X} \subset X$ and $\tilde{Y} \subset Y$ total-measure subset with respect to any ergodically supported measure such that (\tilde{X}, T) and (\tilde{Y}, S) are topologically conjugated.*

Proof. It is a consequence of Theorem 1.2 in [7] and the previous theorem. □

Now, we are able to get sufficient conditions on a dynamical system that allow to define a topological weak quasi-group operation for which the dynamical system map becomes an automorphism.

Theorem 3.4. *Let (X, T) be a topological dynamical system, and assume it is invertible, expansive, ergodically supported with odd ergodic*

period, and has the shadowing property. If $\mathbf{h}(T) = \log(N)$ for some positive integer N , then there exists a topological weak quasi-group operation \bullet with respect to any ergodically supported measure of (X, T) , for which T is an automorphism.

Proof. Let B be the ergodic period of (X, T) . Define the dynamical system (Y, S) as $Y := \{0, 1, \dots, N-1\}^{\mathbb{Z}} \times \{0, 1, \dots, B-1\}$ and $S := \sigma \times s$ the map where $\sigma : \{0, 1, \dots, N-1\}^{\mathbb{Z}} \rightarrow \{0, 1, \dots, N-1\}^{\mathbb{Z}}$ is the shift map and $s : \{0, 1, \dots, B-1\} \rightarrow \{0, 1, \dots, B-1\}$ is the permutation defined by $s(i) := i + 1 \pmod{B}$.

Since (X, T) and (Y, S) are invertible, expansive, ergodically supported, have the shadowing property, and have the same topological entropy and the same ergodic period, then from Corollary 3.3 there exist $\tilde{X} \subset X$ and $\tilde{Y} \subset Y$ total-measure subsets with respect to any ergodically supported measure and $\varphi : \tilde{X} \rightarrow \tilde{Y}$ which is topological conjugacy between (\tilde{X}, T) and (\tilde{Y}, S) .

Define on Y the quasi-group operation $*$ given by

$$\left((x_i)_{i \in \mathbb{Z}}, a\right) * \left((y_i)_{i \in \mathbb{Z}}, b\right) := \left((x_i \tilde{*} y_i)_{i \in \mathbb{Z}}, \lambda(a \tilde{+} b)\right), \quad (1)$$

where $\tilde{*}$ is any quasi-group operation on $\{0, 1, \dots, N-1\}$, $\lambda := (B+1)/2$ and $\tilde{+}$ is the sum \pmod{B} . It is straightforward that the shift map σ is an automorphism for $\tilde{*}$ and, from Lemma 2.1, the map s is an automorphism for $\tilde{+}$. Thus, S is an automorphism for $*$. Furthermore, since $\tilde{*}$ is a 1-block operation (see [6]) and $\tilde{+}$ is continuous for the power set topology on $\{0, \dots, B-1\}$, then $*$ is a topological quasi-group operation.

Denote the by $\Theta : Y \times Y \rightarrow Y$ the map given by $\Theta(\mathbf{x}, \mathbf{y}) = \mathbf{x} * \mathbf{y}$, for any $\mathbf{x}, \mathbf{y} \in Y$. Since Θ is continuous, the set $\Theta^{-1}(\tilde{Y}) \subseteq Y \times Y$ has total measure subset with respect to the product measure of any ergodically supported measure of Y in the product space $Y \times Y$.

Thus,

$$\Lambda := (\tilde{Y} \times \tilde{Y}) \cap \Theta^{-1}(\tilde{Y})$$

is also a total measure subset with respect to the product measure of any ergodically supported measure of Y in the product space $Y \times Y$. Note that, Λ is the set of all pair of points of $\tilde{Y} \times \tilde{Y}$ for what the product by $*$ is a point lying in \tilde{Y} . Furthermore, since Θ commutes with the maps $S \times S$ and S , and \tilde{Y} is S -invariant, we get that Λ is $S \times S$ -invariant.

Define $\Omega \subseteq \tilde{X} \times \tilde{X}$ by

$$\Omega := (\varphi \times \varphi)^{-1}(\Lambda).$$

Since Λ is a total measure subset with respect to the product measure of any ergodically supported measure of Y in the product space $Y \times Y$, and $\varphi \times \varphi : \tilde{X} \times \tilde{X} \rightarrow \tilde{Y} \times \tilde{Y}$ is a homeomorphism, then Ω is a total measure subset with respect to the product measure of any ergodically supported measure of X in the product space $X \times X$. Hence, for any pair $(x, y) \in \Omega$ we can define the quasi-group operation \bullet given by

$$x \bullet y := \varphi^{-1}(\varphi(x) * \varphi(y)).$$

Note that \bullet is well define. In fact, since $(x, y) \in \Omega$, then $(\varphi(x), \varphi(y)) \in \Lambda$. Therefore $(\varphi(x) * \varphi(y)) \in \tilde{Y}$ and $\varphi^{-1}(\varphi(x) * \varphi(y)) \in \tilde{X}$.

Furthermore, for any $(x, y) \in \Omega$ it follows that

$$\begin{aligned} T(x \bullet y) &= T(\varphi^{-1}(\varphi(x) * \varphi(y))) = \varphi^{-1}(S(\varphi(x) * \varphi(y))) \\ &= \varphi^{-1}(S(\varphi(x)) * S(\varphi(y))) = \varphi^{-1}(\varphi(T(x)) * \varphi(T(y))) = T(x) \bullet T(y). \end{aligned}$$

□

Note that we can get the weak quasi-group operation in the previous theorem as being associative if, and only if, (X, T) is ergodically aperiodic. In fact, if (X, T) is not ergodically aperiodic then (Y, S) is also not ergodically aperiodic. On the other hand, if S would be an expansive and ergodic automorphism for some group operation, then from Theorem 1(iv) in [3] we could get that (Y, T) is topologically conjugated to a full shift, which would contradict that (Y, S) was not ergodically aperiodic.

The next theorem given a standard decomposition for the dynamics of maps which topological entropy $\log(N)$ with N integer.

Theorem 3.5. *Let (X, T) be an invertible, expansive, ergodically-supported and aperiodic topological dynamical system with the shadowing property. Suppose $\mathbf{h}(T) = \log(N)$, where $N = p_1 \cdots p_q$ is the prime decomposition of $N \in \mathbb{N}$. Then, there exist an Abelian topological weak group operation \bullet on X for which T is an automorphism,*

and T -invariant subsets $X_j \subseteq X$ for $j = 1, \dots, q$, such that almost any $\mathbf{x} \in X$ can be written in a unique way as $\mathbf{x} = \sum_{j=1}^q \mathbf{x}_j := \mathbf{x}_1 \bullet \dots \bullet \mathbf{x}_q$, with $\mathbf{x}_j \in X_j$.

Proof. Define Abelian topological weak group (X, \bullet) by the procedure given in Theorem 3.4, taking

• as the projection on $\tilde{X} \subset X$ by $\varphi : \tilde{X} \rightarrow \tilde{Y}$ of some Abelian group operation $*$ defined on $Y = \mathcal{A}^{\mathbb{Z}}$, where $\#\mathcal{A} = N$. We recall that $*$ is canonically induced from a group operation on \mathcal{A} (which we will denote as $\tilde{*}$ as in (1)). Since $N = p_1 \cdots p_q$, the group $(\mathcal{A}, \tilde{*})$ is isomorphic to $\bigoplus_{j=1}^q \mathbb{Z}_{p_j}$. Hence, we can identify each $x \in \mathcal{A}^{\mathbb{Z}}$ with one element $(z_1, \dots, z_q) \in \bigoplus_{j=1}^q (\mathbb{Z}_{p_j})^{\mathbb{Z}}$ and, denoting by e the identity element of any \mathbb{Z}_{p_i} , we can define $Y_j := \{x \in \mathcal{A}^{\mathbb{Z}} : x \equiv (z_1, \dots, z_q), \text{ where } z_j \in (\mathbb{Z}_{p_j})^{\mathbb{Z}} \text{ and } z_i \text{ is the identity element of } (\mathbb{Z}_{p_i})^{\mathbb{Z}} \text{ for } i \neq j\}$ and $X_j := \varphi^{-1}(Y_j \cap \tilde{Y})$ satisfy the theorem. □

Note that the sets X_j in the above theorem have null measure. However the theorem says that the dynamical behavior of those maps can be decomposed (except for a universally null set) as the action of the map on the subsets X_j . In this way, we can consider the family set $\{X_j\}_{j=1, \dots, q}$, as a type of standard decomposition of X for the dynamics of T in an analogous way to the decomposition of the action of linear maps in terms of their eigenspaces.

4 Final discussion

Note that the converse statement of Theorem 3.4 would allow to extend the results of [4] for the more general case where $T : X \rightarrow X$ is an expansive ergodically supported map with the shadowing property and an automorphism for some topological quasi-group operation. However, seems to be not direct that the existence of a topological (weak) quasi-group operation on X for which T is an automorphism implies that $\mathbf{h}(T) = \log(N)$. Note that, since any topological (weak) quasi-group operation on X induces a topological weak quasi-group operation on the symbolic representation of X , then we should be able to assure that the existence of weak quasi-group operation on a

shift space product a finite set implies that this shift space has topological entropy $\log(N)$. It would be some kind of extension of the consequences of Theorem 1 in [3] (for groups) and Theorem 4.25 in [6] (for quasi groups).

It would also be interesting to study the case of non-invertible maps. In such a case we cannot apply Theorem 3.3 since the projections $p_T : X^T \rightarrow X$ and $p_S : Y^S \rightarrow Y$ are not invertible. However, there are several examples of non-invertible maps which are topologically conjugated to shift spaces outside of a universally null set (for instance, the maps on the unit interval with the form $T(x) = Mx \pmod{M}$), and for which we could apply Theorem 3.4. However, it remains open if it is possible to obtain a version of Theorem 3.4 for general non-invertible maps.

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